

# NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 4

Fall 2023

## Problem 4

In the Fibonacci sequence,  $1, 1, 2, 3, 5, \dots$  each term after the second is the sum of the previous two terms. How many of the first 100 terms of the Fibonacci sequence are odd?

**Solution:** Consider the parity of the Fibonacci numbers:

$odd, odd, even, odd, odd, even, odd, odd, even, \dots$

The pattern  $odd, odd, even$  repeats itself. This is indeed the case because

$$\begin{aligned} odd + odd &= even, \\ odd + even &= odd, \\ even + odd &= odd. \end{aligned}$$

Therefore, we know the first 99 terms of the Fibonacci sequence have 66 odd numbers and 33 even numbers. Moreover, the  $100^{th}$  number should be an odd number. In total, there are 67 odd numbers. ■

**Remark:** We observed here that the sequence  $\{F_n \pmod{2}\}$  is periodic, where it repeats every 3 terms. In fact, the sequence  $\{F_n \pmod{m}\}$  is periodic for any  $m$ . The length of this period is called the Pisano period, denoted by  $\pi(m)$ . In this question, we observed that  $\pi(2) = 3$ . This turns out to be the only odd Pisano number. You can read more about it here: [https://en.wikipedia.org/wiki/Pisano\\_period](https://en.wikipedia.org/wiki/Pisano_period)